made by first specifying the initial position and the time of intercept and then applying the velocity computed by either the linear solution or the second-order solution. The resulting initial conditions are then propagated by numerical integration of the full, nonlinear equations of motion (Eqs. (1–6) in Ref. 15) to the specified time of intercept in order to calculate the miss distance incurred by the approximations.

Figure 1 shows the miss distance as a function of τ_f for the initial condition $x_i = y_i = z_i = 0.001$, where the variables x, y, and z are the nondimensional rectangular components of the relative position in the radial, in-track, and cross-track directions, respectively. The initial conditions in terms of the relative spherical coordinates may be obtained using the transformations between rectangular and spherical coordinates given in Ref. 15. The first-order solution is shown by the dashed curve, while the second-order solution is shown by the solid curve. The second-order approach shows an order-of-magnitude improvement over the first-order solution.

Conclusions

This Note has developed a solution of Lambert's problem, using a second-order approximation of the equations of motion, referenced to a point on a nearby circular orbit. The solution was obtained from a multiple-scale solution of Kepler's problem in which the initial velocity variables were determined by an algebraic perturbation method, given the initial and final positions. The results show a significant improvement over the previous solution derived from a first-order approximation of the equations of motion. The advantages of such a method are an improved numerical accuracy as well as the coupling of in-plane and out-of-plane motion captured at second order.

Acknowledgment

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Sequential Computation of Total Least-Squares Parameter Estimates

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Introduction

THE most common method to solve an overdetermined set of linear equations is the least-squares estimator (LS). The numerical simplicity of the LS regression estimator and the availability of recursive algorithms are probably the prime reasons behind its extreme proliferation. Although LS regression only acknowledges disturbances in the dependent variables, it is often applied to cases where not only the system's output but also the independent explanatory variables are affected by uncertainties. This applies to many aerospace applications, for example, in the equation error approach to aerodynamic model development and validation from flight-test data. Here, both dependent and independent variables are directly or indirectly derived from measurements on the vehicle states and inputs and are thus corrupted by errors. However, the noise that affects the measurements on the explanatory variables is not properly addressed by an LS estimator.

The counterpart of the least-squares estimator that correctly handles the error-in-variables problem is the total-least-squares estimator (TLS).[1] Instead of minimizing the sum of squares of residuals on only the response variable, it seeks to minimize the sum of squares of residuals on all of the variables in the equation. Unfortunately, TLS estimators do not share the desirable computational properties of the ordinary LS estimators. A recursive algorithm that directly propagates a TLS estimate over the incoming measurements is not available.[2] Total least-squares parameter estimates are found by computing the singular value decomposition (SVD) of the compound matrix of explanatory and explained variables.[3] Because the size of this matrix is directly related to the number of measurements, computation of a TLS estimate can be problematic for large sets of measurements. Although no direct recursive algorithms are known, sequential techniques that determine an updated SVD by means of another singular value decomposition are available[4]; the latter however is of a constant dimension that is related to the number of model parameters and not to the number of measurements.

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Being part of most robust and adaptive control systems, least-squares estimators are used in an environment where computational effort and manageability of data are of great importance. Efficient recursive or sequential algorithms are therefore mandatory. At the same time, the context of measured data that corrupts both dependent and independent variables constitutes a strong preference for total least-squares estimators. This Note presents a brief analysis of the TLS problem as it is typically encountered during parameter estimation for aerospace dynamic models. Based on this analysis, an efficient method for the sequential computation of the TLS estimate is proposed.

Preliminaries

The ordinary least-squares problem deals with the determination of the vector $\boldsymbol{x} \in \mathbb{R}^n$ that minimizes $\|A\boldsymbol{x} - \boldsymbol{b}\|_2$, in which the matrix of independent variables $A \in \mathbb{R}^{m\times n}$ and the vector of dependent variables $\boldsymbol{b} \in \mathbb{R}^m$ are the known elements in the overdetermined set of equations $\boldsymbol{b} \approx A\boldsymbol{x}$. If rank$(A)$ equals the dimension of the parameter vector n, the least-squares problem has the unique solution $\boldsymbol{x}_{\mathrm{LS}} = (A^T A)^{-1} A^T \boldsymbol{b}$ (Refs. 1 and 3). The recursive least-squares algorithm computes the solution to the LS problem for $A_m^T = [A_{m-1}^T, \boldsymbol{a}_m^T]$ and $\boldsymbol{b}_m^T = [\boldsymbol{b}_{m-1}^T, b_m]$ from the solution for the case $A_{m-1}, \boldsymbol{b}_{m-1}$. If the matrix $A_m^T A_m = A_{m-1}^T A_{m-1} + \boldsymbol{a}_m^T \boldsymbol{a}_m$ is written as $P_{m-1}^{-1} + \boldsymbol{a}_m^T I \boldsymbol{a}_m$, the matrix inversion lemma can be used to yield

$$\left(A_m^T A_m\right)^{-1} = P_m = P_{m-1} - \frac{P_{m-1}\boldsymbol{a}_m^T \boldsymbol{a}_m P_{m-1}}{1 + \boldsymbol{a}_m P_{m-1} \boldsymbol{a}_m^T} \tag{1}$$

in which the remaining inverse is scalar. Setting $\boldsymbol{k} = (P_{m-1}\boldsymbol{a}_m^T)/(1 + \boldsymbol{a}_m P_{m-1}\boldsymbol{a}_m^T)$ and using Eq. (1), the recursive least-squares estimator consists of the following two steps after the computation of $\boldsymbol{k}$ (Ref. 5):

$$P_m = P_{m-1} - \boldsymbol{k}\boldsymbol{a}_m P_{m-1}, \qquad \boldsymbol{x}_m = \boldsymbol{x}_{m-1} + \boldsymbol{k}(b_m - \boldsymbol{a}_m \boldsymbol{x}_{m-1}) \tag{2}$$

Because the matrix A contains the set of row vectors of explanatory variables—one for each measurement—and the rank of a matrix equals its number of independent row vectors, rank(A) cannot decrease when a new measurement is added. Once enough independent measurements have been collected, the matrix $A^T A$ therefore cannot become rank deficient again, although its condition might deteriorate. This ensures succesful propagation of the matrix P, a property that will prove useful for sequential TLS estimation as well.

The total least-squares solution for the overdetermined set $\boldsymbol{b} \approx A\boldsymbol{x}$ is the vector that satisfies the approximate set of compatible equations $\boldsymbol{b}' = A'\boldsymbol{x}_{\mathrm{TLS}}$, for which the Frobenius norm $\|[A, \boldsymbol{b}] - [A'\boldsymbol{b}']\|_F$ is minimal.[1] If $U\Sigma V^T$ is the singular value decomposition of $[A, \boldsymbol{b}]$ where $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})$ contains the ordered set of real singular values for which $\sigma_i \geq \sigma_{i+1}$, then the closest approximate set of rank n is $U\Sigma' V^T$ with $\Sigma' = \mathrm{diag}(\sigma_1, \ldots, \sigma_n, 0)$. The desired solution $\boldsymbol{x}_{\mathrm{TLS}}$ must then satisfy $U\Sigma' V^T [\boldsymbol{x}_{\mathrm{TLS}}^T, -1]^T = 0$. Hence, the vector $[\boldsymbol{x}_{\mathrm{TLS}}^T, -1]^T$ is part of the kernel of $U\Sigma' V^T$ and must be perpendicular to the first n column vectors of V. As V is orthonormal, the desired vector equals the last column vector of V.

Sequential Total Least Squares

The singular values of a matrix C are the square roots of the eigenvalues of the matrix $C^T C$; the columns of the matrix of right singular vectors V are the corresponding eigenvectors of $C^T C$. The TLS problem is thus reduced to finding the eigenvector that is associated with the smallest eigenvalue of $[A, \boldsymbol{b}]^T [A, \boldsymbol{b}]$. Computation of $C^T C$ is usually strongly discouraged because of numerical inaccuracies.[1,3] When the original matrix is ill conditioned, the product $C^T C$ can become singular as a result of finite-precision computations. However, examples of such matrices are highly academic. Ill conditioning in a system identification application caused by insufficient excitation does not play a role here. As was noted before, a full-rank matrix of variables cannot become rank deficient again. Erroneous singularity of the matrix $[A, \boldsymbol{b}]^T [A, \boldsymbol{b}]$ can only occur when a newly added row of measurements contains solely elements that lead to underflow of all previous measurements.

Assuming measurement errors (spikes) have been removed, this is not a realistic scenario. Additionally, if such measurements would occur the ill conditioning of the matrix would also lead to unreliable parameter estimates if computation takes place with infinite precision.

The eigenvector that is associated with the smallest eigenvalue of an invertible matrix equals the eigenvector for the largest eigenvalue of the matrix' inverse. The power method[3] is based on the characteristic that $\lim_{k\to\infty} A^k \boldsymbol{x}$ converges to a multiple of the dominant eigenvector of A that is not perpendicular to the initial $\boldsymbol{x}$; the dominant eigenvector is the one associated with the largest eigenvalue. Application of the power method to the inverse of a matrix therefore produces a series of vectors that converge to the eigenvector for the smallest eigenvalue of the original matrix. A TLS estimate is thus most easily found by applying the power method to $([A, \boldsymbol{b}]^T [A, \boldsymbol{b}])^{-1}$.

At this point, a sequential algorithm for computing TLS estimates can be formulated on the basis of propagation of the matrix $P = ([A, \boldsymbol{b}]^T [A, \boldsymbol{b}])^{-1}$, similar to role of the matrix P in recursive ordinary least squares. Because the power method computes the parameter estimate from the propagated matrix directly, the estimate itself is not used in the recursion. Hence, the complete TLS propagation consists only of

$$P_m = P_{m-1} - \frac{\boldsymbol{p}^T \boldsymbol{p}}{1 + \boldsymbol{p}[\boldsymbol{a}_m, b_m]^T} \tag{3}$$

with $\boldsymbol{p} = [\boldsymbol{a}_m, b_m] P_{m-1}$. If the actual estimate is required, it can be computed by updating the eigenvector estimate $\boldsymbol{v}$ in the iteration

$$\boldsymbol{v}_{k+1} = P(\boldsymbol{v}_k / v_{k,n+1}) \tag{4}$$

In Eq. (4), $v_{k,n+1}$ denotes the $(n+1)$th element of the vector $\boldsymbol{v}_k$. By dividing the vector by its last element, an explosion of the iterated vector and potential numerical problems are avoided. Because eigenvectors can arbitrarily be scaled, this does not influence the iteration itself. Instead, because the last element of the vector is repeatedly scaled to 1, $\boldsymbol{v}_{k+1,n+1}$ converges to the largest eigenvalue of P and can be used as a convergence requirement for the iteration: The dominant eigenvector is found when the difference between $v_{k,n+1}$ and $v_{k+1,n+1}$ drops below a preset convergence requirement. By choosing $\boldsymbol{v}_0 = [0, \ldots, 0, 1]^T$, it is guaranteed that the vector has a component along the desired eigenvector. Because the converged vector can be used as the starting point for a later iteration when P has been updated, $\boldsymbol{v}$ needs only be initilized once. Finally, the actual parameter estimate is obtained from the eigenvector estimate:

$$\boldsymbol{x}_{\mathrm{TLS}} = -v_{1:n}/v_{n+1} \tag{5}$$

Computational Effort

The proposed algorithm was implemented in MATLAB® to compare its performance with that of the existing recursive total least-squares methods based on the singular value decomposition[4] and a recursive ordinary least-squares algorithm. (The MATLAB script files are available for download from the author's web site at http://www.soijer.de.) Figure 1 shows the computational effort for the various methods. Artificial experimental data are analyzed for 100 time steps, using a parameter vector of varying order between 2 and 25. The sequential total least-squares method requires more operations than ordinary least squares. The main contributor to this is the iteration (4) that is required to find the parameter estimate; it was terminated when the dominant eigenvalue reached a precision of $1 \cdot 10^{-8}$. A considerable improvement is found with respect to SVD-based recursive total least squares, starting at one order of magnitude and increasing with growing parameter vectors. In the figure, the existing SVD-based method is indicated as recursive TLS; the method that is proposed in this Note is labeled sequential TLS.

When the iterative parameter estimation is left out and only the matrix inverse of variables is propagated, the sequential TLS method can be applied with less effort than recursive ordinary least squares for any vector of order 5 or larger. Because the sequential TLS

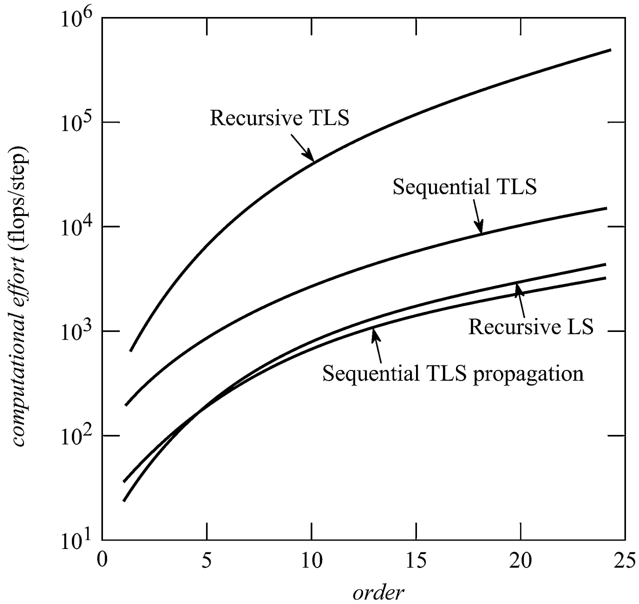

Fig. 1 Averaged computational effort for LS and TLS estimators.

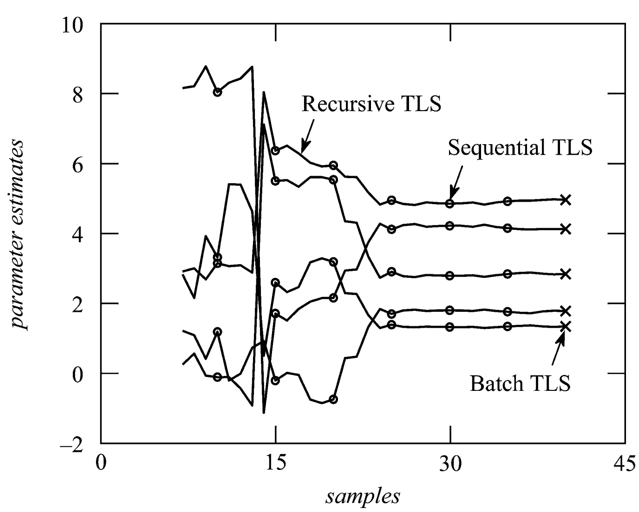

Fig. 2 Resulting estimates for five parameters from 40 samples.

method does not depend on the parameter estimate for its propagation, parameter estimates can be computed at a reduced rate or by another process without loosing numerical accuracy. This is illustrated in Fig. 2, where the estimates for a five-element parameter vector are shown for the various TLS methods. The crosses at the right-hand side mark the results from an SVD-based batch TLS estimation that includes 40 samples; the solid lines indicate the recursive TLS solution that approaches the batch solution with each additional sample that is included. The circles indicate selected estimates with the proposed sequential method, in this case once every five samples. It is clear that sequential TLS provides the same parameter estimates as recursive TLS and that both converge towards the batch solution.

Conclusions

The application of total least squares to typical aerospace parameter estimation problems was briefly discussed. The commonly mentioned threat of information loss by reducing the variables matrix to its inner square was analyzed and found harmless to applications where a series of measurements arrives with time. Together with the notion that instead of singular values, only the smallest eigenvector of the inner square matrix is required to compute TLS estimates this led to the presentation of a computationally superior sequential TLS method.

The suggested method satisfies all of the requirements on an estimator for real-time applications: Its computational demand for each step is independent of the number of preceding measurements and memory requirements are constant. Propagation of the inverted inner square matrix with each arriving measurement does not depend on computation of the actual parameter estimate; without it, the number of operations per step is deterministic and smaller than that for the recursive ordinary least-squares estimator.

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Optimal Interplanetary Trajectories Using Constant Radial Thrust and Gravitational Assists

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Nomenclature

a_r	=	radial thrust acceleration
dr/dt	=	radial rate
$d\theta/dt$	=	angular rate
$E(k, \phi)$	=	elliptic integral of the second kind
$F(k, \phi)$	=	elliptic integral of the first kind
h	=	angular momentum
K_{ACO}	=	total energy after cutoff
K_{BCO}	=	total energy before cutoff
p	=	semilatus rectum
r, θ	=	polar coordinates
r_{CO}, v_{CO}	=	cutoff radius and velocity

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